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Stability and Positive Supermartingales*

by

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The problem considered here is the stability of solutions of non-linear difference-equations containing random elements. Guided by the Liapounov theory for deterministic systems we introduce the concepts of a random equilibrium point and stability of a random solution in probability as well as an almost everywhere or almost sure asymptotic stability. These concepts seem the natural counterparts for random systems of the Liapounov theory with positive supermartingales (see [3]) corresponding to Liapounov functions.

The results obtained are roughly as follows. A sufficient condition for stability in probability is the existence of a positive definite continuous function which is a supermartingale along the solutions and for asymptotic stability almost everywhere the existence of a decreasing positive supermartingale. Showing the counterpart of the Massera theorem (see [5]), namely that the existence of a Liapounov function is a necessary condition for an appropriate type of stability, seems as yet elusive. In particular we give examples of the application of the above theorems to random difference equations. Of course our results generalize to random differential equations. However the details tend to obscure the ideas and the generalization is not given here.

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I. Definitions and theorems.

We consider the following random system of equations

$$x_n = f(x_{n-1}, r_{n-1}) \quad (1.1)$$

and x_0 a given random vector variable, where f is a continuous real valued vector function and r_{n-1} a sequence of random quantities. It will be assumed that there exist an a.e. unique random variable x_e satisfying

$$x_e = f(x_e, r_n) \quad \text{for all } n \geq 0 \quad (1.2)$$

and x_e will be called an equilibrium point. In the sequel it will be necessary to distinguish two situations; namely

- 1) x_e and x_0 are almost everywhere constant a.e. $P^{(1)}$.
- 2) Either or both x_0 and x_e are not a.e. constant P .

The first case will be called degenerate. Now we define a class of allowable initial conditions in the degenerate and nondegenerate cases;

Definition 1. $C(M, x_e)$ is the class of all initial random variables x_1 such that $\text{ess sup} \|x_1 - x_e\| \leq M$ ⁽¹⁾ in the nondegenerate case and all a.e. constant random variables in the degenerate case.

Having defined an admissible class of initial conditions $C(M, x_e)$ we introduce the following concepts of stability and asymptotic stability of an equilibrium point of (1.1); denoting by $\varphi(n, x_1)$ the solution of (1.1) with initial condition x_1 .

⁽¹⁾ $\| \cdot \|$ denotes Euclidean norm, and P the probability measure.

Definition 2. An equilibrium point x_e of (1.1) is stable relative to $C(M, x_e)$ iff for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every $x_1 \in C(M, x_e)$ satisfying $P(\|x_1 - x_e\| \geq \delta) < \delta$ then $P(\|\phi(n, x_1) - x_e\| \geq \epsilon) < \epsilon$ for all $n \geq 0$.

Definition 3. An equilibrium point x_e of (1.1) is asymptotically stable relative to $C(M, x_e)$ iff it is stable relative to $C(M, x_e)$ and for all $x_1 \in C(M, x_e)$

$$\|\phi(n, x_1) - x_e\| \rightarrow 0 \text{ a.e. } P \text{ as } n \rightarrow \infty.$$

In the degenerate case relative to $C(M, x_e)$ will be deleted from reference to these stabilities. $C(M, x_e)$ is in fact the set of random variables close to x_e in an almost everywhere sense.

In order to state our stability theorems it will be necessary to introduce the concept of a supermartingale.

Definition 4. A sequence of random variables y_n and \mathcal{F}_n sigma fields form a supermartingale sequence iff \mathcal{F}_n is the minimal sigma field over (y_1, \dots, y_n) and

$$E(y_n | \mathcal{F}_{n-1}) \leq y_{n-1} \text{ a.e. } P,$$

where $E(\cdot | \mathcal{F}_n)$ is the conditional expectation [see [2] Chapter VIII.]

Now the following theorem gives a sufficient condition for stability.

Theorem 1. Suppose there exists a positive definite continuous function V continuous at ∞ such that $V(0) = 0$ and the sequence $V(\phi(n, x_1) - x_e)$ for all $x_1 \in C(M, x_e)$ is a supermartingale with x_e an equilibrium point of (1.1). Then x_e is stable relative to $C(M, x_e)$.

Proof. There exist α and β continuous positive non-decreasing functions such that

$$\alpha(\|x - x_e\|) \leq V(x - x_e) \leq \beta(\|x - x_e\|) \quad (1.3)$$

[see [1] proof of corollary [1.2] for the construction]. Now for any nondecreasing positive function (Borel function) g and random variable x it is known that

$$\frac{Eg(x) - g(a)}{\text{a.s. sup } g(x)} \leq P[x \geq a] \leq \frac{Eg(x)}{g(a)} \quad (1.4)$$

[see for example [3] page 157]. Now for any $\epsilon > 0$ choose δ sufficiently small so that

$$\epsilon \alpha(\epsilon) > \delta \beta(M) + \beta(\delta). \quad (1.5)$$

For any $x_1 \in C(M, x_e)$ chosen so that

$$\delta > P(\|x_1 - x_e\| \geq \delta)$$

it follows from (1.4) that

$$P(\|x_1 - x_e\| \geq \delta) \geq \frac{\delta \beta(\|x_1 - x_e\|) - \beta(\delta)}{\beta(M)} \geq \frac{EV(x_1 - x_e) - \beta(\delta)}{\beta(M)}. \quad (1.6)$$

Consequently (1.6) and the supermartingale property implies

$$\begin{aligned} \delta B(M) + \beta(\delta) &\geq EV(x_1 - x_e) \geq EV(\varphi(n, x_1) - x_e) \\ &\geq E\alpha(\|\varphi(n, x_1) - x_e\|) \geq \alpha(\epsilon)P(\|\varphi(n, x_1) - x_e\| \geq \epsilon) \end{aligned}$$

and using (1.4) or with (1.5) that

$$P(\|\varphi(n, x_1) - x_e\| \geq \epsilon) < \epsilon \text{ independently of } n. \quad (1.7)$$

Regarding asymptotic stability the following theorem holds:

Theorem 2. Suppose the assumptions of the previous theorem hold and further that there exists a continuous function γ such that $\gamma(0) = 0$ and

$$EV(\varphi(n, x_1) - x_e) - V(\varphi(n-1, x_1) - x_e) \leq -\gamma(\|\varphi(n-1, x_1) - x_e\|) < 0$$

for all $x_1 \in C(M, x_e)$. Then x_e an equilibrium solution of (1.1) is asymptotically stable relative to $C(M, x_e)$.

Proof. Let $N_n = \|\varphi(n, x_1) - x_e\|$. Now letting V_n denote $V(\varphi(n, x_1) - x_e)$ it follows that

$$EV_{n+1} - EV_0 \leq \sum_{i=0}^n -E\gamma(N_i) \text{ for all } n \geq 0$$

or

$$0 \leq \sum_{i=0}^n E\gamma(N_i) \leq EV_0 \leq \beta(M) < \infty. \quad (1.8)$$

But (1.8) implies since $\gamma > 0$ that

$$E r(N_n) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (1.9)$$

As $r(N_n) \rightarrow 0$ in probability pick a subsequence n_v , so that $r(N_{n_v}) \rightarrow 0$ almost surely. Then as r is continuous and vanishes only at zero $N_{n_v} \rightarrow 0$ almost surely. Doob's semi-martingale convergence theorem (see [3] page 324) applied to $\{-V_n\}$ implies there exists a V such that

$$V_n \xrightarrow[L^1]{a.e.} V \text{ as } EV_1 \leq \beta(M). \quad (2.0)$$

However using the α and β introduced in Theorem 1

$$\beta(N_n) \geq V_n \geq \alpha(N_n) \quad (2.1)$$

so that taking the limit of the left side of (2.1) along n_v implies

$$\beta(0) \geq V \geq 0.$$

Hence $V = 0$ a.e. or from (2.1) $0 = \alpha(\lim N_n)$ since α is continuous and

$$\lim \|\phi(n, x_1) - x_e\| = 0 \text{ a.e.P.}$$

Now we apply the previous theorems to the following examples.

Example 1.

Let $x_n = A_{n-1}x_{n-1}$ where A_{n-1} are independent identically distributed random matrices and suppose there exists a positive definite matrix B so that $EA_n'BA_n - B$ is negative definite. Then 0 is asymptotically

stable relative to $C(M, 0)$ for any M , since with $V(x) = x'Bx$ Theorem 2 is seen to apply. Further, for any initial vector random variable x_0 such that $Ex_0'Bx_0 < \infty$, $x_n \rightarrow 0$ a.e.P and L^2 as can be seen by examination of the proof of Theorem 2. L^2 convergence of the system is of course well known under a tensor product condition corresponding to the above condition (see [7]).

Example 2.

We consider the following system

$$x_n = \delta_{n-1}x_{n-1} + \alpha_{n-1}y_{n-1}$$

$$y_n = \beta_{n-1}y_{n-1} - \epsilon_{n-1} \left(\frac{1 - x_{n-1}^2}{1 + x_{n-1}^2} \right) y_{n-1} - \gamma_{n-1}x_{n-1}$$

where $(\alpha_n, \beta_n, \gamma_n, \delta_n, \epsilon_n)$ are pairwise independent mean zero random variables and for each n , α_n etc. are identically distributed. Now if $E(\delta_n^2 + \gamma_n^2) < 1$ and $E(\alpha_n^2 + \beta_n^2 + \epsilon_n^2) < 1$ then the equilibrium point $(0, 0)$ is seen to be asymptotically stable by Theorem 2 using the function $V(x, y) = x^2 + y^2$.

In [4] a rather similar definition of stability in probability is given but the results are weaker, and almost sure stability is not considered. We mention that a result of the type of [1], section 9, can easily be obtained for the stability of an optimized stochastic control system (see [6]).

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